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COVERING NUMBERS OF MANIFOLDS AND CRITICAL POINTS OF A MORSE FUNCTION†

BY

ALBERTO CAVICCHIOLI

Istituto di Matematica, Universita della Basilicata, Via N. Sauro n. 85, 85100 Potenza, Italy

ABSTRACT

For a closed connected triangulated n-manifold M, we study some numerical invariants (named *category* and *covering numbers*) of M which are strictly related to the topological structure of M. We complete the classical results of 3-manifold topology and then we prove some characterization theorems in higher dimensions. Finally some applications are given about the minimal number of critical points (resp. values) of Morse functions defined on a closed connected smooth n-manifold.

1. Notations

Let N_n be the set $\{0, 1, \ldots, n\}$ and $N_n^0 = N_n - \{0\}$. For a set A, the symbol card(A) means the cardinality of A. If X is a topological space, then $H_q(X)$ (resp. $\tilde{H}_q(X)$) denotes the q-th integral (resp. reduced) homology group of X for each integer $q \geqq 0$. Further $H_q(X; Z_2)$ (resp. $\tilde{H}_q(X; Z_2)$) represents the q-th (resp. reduced) homology group of X with mod 2 coefficients and $\Pi_q(X)$ means the q-th ($q \geqq 1$) homotopy group of X.

Given a group G, the direct sum $G \oplus \cdots \oplus G$ (p times, $p \geqq 1$) will simply be written as pG.

Throughout the paper we will work in the piecewise linear (PL) category in the sense of [Gl] and [RS]. The prefix PL will always be omitted. Manifolds will be compact and connected. Given an n-manifold M^n, ∂M and Int(M) denote the *boundary* and the *interior* of M respectively. M is said to be *closed* if ∂M is

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empty. If P is a compact polyhedron in M, then $N(P; M)$ represents a *regular neighbourhood* of P in M (see [RS, p. 33]). The *interior* and the *closure* of P in M will be denoted by $\mathrm{Int}(P; M)$ and $\mathrm{cl}(P; M)$ respectively. P is said to be *bicollared* in M if there exists an embedding $f: P \times [-1, 1] \to M$ such that $f(x, 0) = x$ for every point $x \in P$. An *n-ball* B^n is an n-manifold homeomorphic to a standard n-simplex. The *n-sphere* S^n is the boundary of an $(n+1)$-ball. For $n \geqq 2$, the symbol $S^1 \otimes S^{n-1}$ (resp. $S^1 \otimes B^{n-1}$) means either the topological product $S^1 \times S^{n-1}$ (resp. $S^1 \times B^{n-1}$) or the twisted S^{n-1} (resp. B^{n-1}) bundle over S^1. Furthermore $k(S^1 \otimes S^{n-1})$ and $k_\partial(S^1 \otimes B^{n-1})$ denote the connected sum and the boundary connected sum (see [R, p. 39]) of k ($k \geqq 1$) copies of $S^1 \otimes S^{n-1}$ and $S^1 \otimes B^{n-1}$ respectively. If $k = 0$, we set $k(S^1 \otimes S^{n-1}) = S^n$ (n-sphere) and $k_\partial(S^1 \otimes B^{n-1}) = B^n$ (n-ball).

A closed n-manifold M is said to have a *topological handle* if M is homeomorphic to a connected sum $M' \# (S^1 \otimes S^{n-1})$ for some closed n-manifold M'. Otherwise M is called a *handle free n-manifold* (see [T2]). We say that M is *trivial* (resp. *non-trivial*) if M is (resp. is not) the n-sphere. A closed n-manifold M is *prime* iff $M = M_1 \# M_2$ implies that M_1 or M_2 is trivial. M is *irreducible* iff each $(n-1)$-sphere in M bounds an n-ball in M. Obviously irreducible n-manifolds are prime and handle free prime n-manifolds are irreducible (see [H], [BD]).

Let now M be an n-manifold with non-empty boundary and let H be an n-ball such that $M \cap H \subseteq \partial M$. If there exists a homeomorphism $f: B^p \times B^{n-p} \to H$ with $f(\partial B^p \times B^{n-p}) = H \cap M$, then we say that H is a *handle of index p attached to M* (see [RS, p. 74]). Obviously the polyhedron $X = M \cup H$ is again an n-manifold. If M is an n-manifold, then a *handle decomposition* of M is a presentation $M = H_0 \cup H_1 \cup \cdots \cup H_t$, where H_0 is an n-ball and H_i ($i \geqq 1$) is a handle attached to $M_{i-1} = \bigcup \{H_j / j \leqq i - 1\}$. Any manifold M admits a handle decomposition, i.e. M can be regarded as constructed from a ball by attaching handles of various indices (see [RS, p. 82]).

As general references for 3-manifold topology see [H], [BD] and [R]. For algebraic topology we refer to [Sp], [Maun], [Sw] and [HW].

2. Covering numbers of manifolds

Let X be a topological space. The *Lusternik–Schnirelmann category* of X, written $\mathrm{cat}(X)$, is the smallest number of contractible closed (resp. open) sets which are required to cover X (see [LS], [F], [Ga], [Gi]). The category is an invariant of the homotopy type, i.e. if X and Y are two topological spaces with the same homotopy type, then $\mathrm{cat}(X) = \mathrm{cat}(Y)$ (see [F, p. 342]). Recall that a closed n-manifold has category 2 iff it is a homotopy n-sphere (see [Si, p. 413]).

Let M^n be a closed n-manifold. Another classical invariant of M is the minimum number $\mathbf{C}(M)$ of n-balls needed to cover M (see [Si], [Mi]). We have the inequalities $2 \leqq \mathrm{cat}(M) \leqq \mathbf{C}(M) \leqq n + 1$ as proved in [Z]. Moreover $\mathbf{C}(M)$ coincides with $\mathrm{cat}(M)$ whenever $\mathrm{cat}(M) \geqq \frac{1}{2}n + 2$ and $n \geqq 4$ (see [Si], prop. 6.2). The main interest about these invariants is given by their close connection with classical problems of manifold topology as the Poincaré and Schoenfliës conjectures (see [KT], [Si], [F], [Z]). Minimal ball coverings of manifolds were also studied by several authors besides the above-mentioned ones (see references of the quoted papers). In the present section we consider ball coverings of closed n-manifolds whose n-balls have nice intersection properties in the sense of the following definitions (see [CG]).

Definition 1. Let M^n be a closed n-manifold and let $\mathbf{B} = \{B_i / i \in I\}$ be a finite set of n-balls in M such that $\bigcup \mathbf{B} = M$. Then $\mathbf{B}$ is called a *P_0-ball covering* (resp. *P_1-ball covering*) of M iff $B_i \cap B_j = \partial B_i \cap \partial B_j$ has $(n-1)$-manifolds (resp. $(n-1)$-balls) as connected components for each $i, j \in I$, $i \neq j$.

The concepts of P_0- and P_1-ball coverings of a manifold M were first introduced in [KT] and [FG] with the names of *ball coverings* and *strong ball coverings* of M respectively.

Definition 2. With the above notation, $\mathbf{B}$ is called a *P_2-ball covering* of M^n iff the components of the intersection

$$\bigcap \{B_j / j \in J\} = \bigcap \{\partial B_j / j \in J\}$$

are $(n - h + 1)$-balls for every subset J of I with $\mathrm{card}(J) = h \in N^0_{n+1} - \{1\}$.

The notion of P_2-ball covering of a manifold was first introduced in [P] with the main goal of finding minimal atlases of balls for closed manifolds. The analogous problem for manifolds with non-empty boundary was solved in [CG].

Now we define the *P_r-covering number*, written $\mathbf{b}_r(M)$, of M $(r \in N_2)$ as the minimum number of n-balls of the P_r-ball coverings of M, i.e.

$$\mathbf{b}_r(M) = \min\{\mathrm{card}(\mathbf{B}) / \mathbf{B} \text{ is a } P_r\text{-ball covering of } M\}.$$

Obviously $2 \leqq \mathrm{cat}(M) \leqq \mathbf{C}(M) \leqq \mathbf{b}_0(M) \leqq \mathbf{b}_1(M) \leqq \mathbf{b}_2(M)$ and $\mathbf{b}_0(M) = 2$ (resp. $\mathbf{b}_1(M) = 3$, $\dim(M) = n \geqq 2$) iff M^n is trivial. The following results are well-known:

PROPOSITION 1.
(a) *If M^n is a closed n-manifold, then* $\mathbf{b}_2(M) = n + 1$ (*see* [P]).
(b) *If M^n is a homology n-sphere and* $\mathbf{b}_0(M) \leqq 3$, *then M^n is trivial* ([KT], prop. 3.2).
If M^n ($n \geqq 5$) is a contractible n-manifold, then $\mathbf{b}_0(M) \leqq 3$ ([KT], p. 138).
(c) *If M^n ($n = p + q$; $p, q \geqq 1$) is a p-sphere bundle over a q-sphere, then* $\mathbf{b}_0(M) \leqq 4$. *Further, if M has a cross section, then* $\mathbf{b}_0(M) \leqq 3$ ([T1], p. 116).
(d) *If M^4 is a homotopy 4-sphere, then* $\mathbf{b}_0(M \# k(S^2 \times S^2)) \leqq 3$ *for some integer* $k \geqq 0$ ([KT], prop. 3.7).
(e) *If M^3 is a closed 3-manifold, then* $\mathbf{b}_0(M) = 3$ *iff M is homeomorphic to* $k(S^1 \otimes S^2)$ *for some integer* $k \geqq 1$ ([KT], th. 4.3; [HM], main theorem).

Let now $\mathbf{B}$ be a P_r-ball covering of a closed n-manifold M ($r \in N_1$). Then the intersection of any h ($3 \leqq h \leqq n + 1$) n-balls of $\mathbf{B}$ has $(n - h + 1)$-manifolds as connected components. The proof can be directly obtained by using the same methods (transversality and general position) described in [Si] (point d, p. 397) for categorial fillings of a manifold.

For a closed one-manifold $M^1 (\simeq S^1)$, the above definitions trivially coincide, whence $\mathrm{cat}(M) = \mathbf{C}(M) = \mathbf{b}_0(M) = \mathbf{b}_1(M) = \mathbf{b}_2(M) = 2$. In dimension two, each P_1-ball covering of a closed surface is just a P_2-ball covering. Thus we have $\mathrm{cat}(M) = \mathbf{C}(M) = \mathbf{b}_0(M) = \mathbf{b}_1(M) = \mathbf{b}_2(M) = 3$ iff M is not trivial (see also [F, p. 354]) and $\mathrm{cat}(S^2) = \mathbf{C}(S^2) = \mathbf{b}_0(S^2) = 2$, $\mathbf{b}_1(S^2) = \mathbf{b}_2(S^2) = 3$. For a closed 3-manifold M^3, the P_r-covering numbers of M are calculated as follows: (1) $\mathbf{b}_0(M) = 2$ (resp. $\mathbf{b}_1(M) = 3$) iff M is trivial; (2) $\mathbf{b}_0(M) = 3$ iff $M \simeq k(S^1 \otimes S^2)$ for some integer $k \geqq 1$ (see prop. 1, point (e)); (3) if M is a non-trivial handle free 3-manifold, then $\mathbf{b}_0(M) = \mathbf{b}_1(M) = \mathbf{b}_2(M) = 4$.

In the present paper we are interested in computing the category and the P_r-covering numbers ($r \in N_1$) for many closed manifolds. We complete the classical results of 3-manifold topology and then we prove some characterization theorems in higher dimensions. Finally some applications are given about the number of critical points (resp. values) of Morse functions.

3. Comparisons

Let M^n be a closed n-manifold ($n \geqq 3$). Now we compare the previous definitions of P_r-ball coverings of M.

PROPOSITION 2. *Let M^3 be a closed 3-manifold with $H_2(M) = 0$. Then each P_0-ball covering $\mathbf{B} = \{B_i / i \in N_3\}$ of M is a P_2-ball covering of M.*

PROOF. Since $H_2(M) = 0$, the manifold M is orientable; otherwise $H_3(M) = 0$ and $H_2(M)$ has non-null torsion. Then the intersection of any two 3-balls of $\mathbf{B}$ has 2-balls as connected components (see [KT], th. 4.4). If (i, j, h, k) is a permutation of N_3, we set $\mathbf{B} = \{B_i, B_j, B_h, B_k\}$. Obviously we have

$$H_3(B_i \cup B_j \cup B_h) \simeq H_3(M - \mathrm{Int}(B_k)) = 0.$$

Now it remains to show that $H_1(B_i \cap B_j \cap B_h) = 0$, i.e. each component of the one-manifold $B_i \cap B_j \cap B_h$ must be an arc. Using the Mayer–Vietoris sequence (written (M, V)-sequence) of the pair $(B_i \cap B_j, B_h)$, we get

$$0 = H_2(B_i \cap B_j) \oplus H_2(B_h) \longrightarrow H_2((B_i \cap B_j) \cup B_h)$$
$$\longrightarrow H_1(B_i \cap B_j \cap B_h) \longrightarrow H_1(B_i \cap B_j) \oplus H_1(B_h) = 0,$$

whence $H_1(B_i \cap B_j \cap B_h) \simeq H_2((B_i \cap B_j) \cup B_h)$. The (M, V)-sequence of the pair $(B_i \cup B_h, B_j \cup B_h)$ gives

$$0 = H_3(B_i \cup B_h) \oplus H_3(B_j \cup B_h) \longrightarrow H_3(B_i \cup B_j \cup B_h)$$
$$\longrightarrow H_2((B_i \cap B_j) \cup B_h) \longrightarrow H_2(B_i \cup B_h) \oplus H_2(B_j \cup B_h) = 0,$$

since $H_q(B_i \cup B_h) \simeq \tilde{H}_{q-1}(B_i \cap B_h)$. This implies that

$$H_1(B_i \cap B_j \cap B_h) \simeq H_2((B_i \cap B_j) \cup B_h) \simeq H_3(B_i \cup B_j \cup B_h) \simeq 0$$

as requested. ∎

PROPOSITION 3. *Let M^3 be a closed 3-manifold with $\Pi_2(M) = 0$. Then each P_0-ball covering $\mathbf{B} = \{B_i / i \in N_3\}$ of M is a P_2-ball covering of M.*

PROOF. The assumption $\Pi_2(M) = 0$ implies that

$$H_2(B_i \cup B_j) \simeq H_1(B_i \cap B_j) = 0$$

for each pair $i, j \in N_3$, $i \neq j$ (see [KT], th. 4.5). The proof also works in the non-orientable case. Thus we have

$$H_1(B_i \cap B_j \cap B_h) \simeq H_2((B_i \cap B_j) \cup B_h) \simeq H_3(B_i \cup B_j \cup B_h)$$

for any subset $\{i, j, h\} \subset N_3$ (see the proof of Proposition 2). Since $H_3(B_i \cup B_j \cup B_h) = 0$ for any M, no matter whether it is orientable or not, the proof is completed. ∎

PROPOSITION 4.

(1) *Let M^3 be a closed prime handle free 3-manifold. Then each P_0-ball covering $\mathbf{B} = \{B_i / i \in N_3\}$ of M is a P_2-ball covering of M.*

(2) *Let M^3 be a closed 3-manifold. Then each P_1-ball covering $\mathbf{B} = \{B_i / i \in N_3\}$ of M is a P_2-ball covering of M.*

PROOF.

(1) Such a manifold M is irreducible, whence $\Pi_2(M) = 0$. Now use Proposition 3.

(2) For any subset $\{i, j, h\} \subset N_3$, we have $H_3(B_i \cup B_j \cup B_h) = 0$ and $H_2(B_i \cup B_j) \simeq H_1(B_i \cap B_j) \simeq 0$. Thus the proof is obtained as shown in Propositions 2 and 3. ∎

EXAMPLE. Now we construct a P_0-ball covering $\mathbf{B} = \{B_i / i \in N_3\}$ of a non-prime closed 3-manifold which is not a P_2-ball covering. Let M_i ($i = 0, 1$) be a closed prime 3-manifold and let $\mathbf{B}_i = \{B_{i,j} / j \in N_3\}$ be a P_2-ball covering of M_i. Let D_i be a 2-ball contained in $B_{i,0} \cap B_{i,1}$ for each $i = 0, 1$. If x_i is a point of $\mathrm{Int}(D_i)$, let $R_i = N(x_i; M_i)$ be a small regular neighbourhood of x_i in M_i ($i = 0, 1$). Let $f: \partial R_0 \to \partial R_1$ be a homeomorphism between two 2-spheres such that $f(\partial R_0 \cap B_{0,j}) = \partial R_1 \cap B_{1,j}$ for each $j = 0, 1$. Then the closed 3-manifold

$$M_2 = (M_0 - \mathrm{Int}(R_0)) \bigcup_f (M_1 - \mathrm{Int}(R_1))$$

is the connected sum $M_0 \# M_1$. Obviously

$$B_j = (B_{0,j} - \mathrm{Int}(R_0)) \bigcup_f (B_{1,j} - \mathrm{Int}(R_1))$$

is a 3-ball in M_2 for each $j = 0, 1$. By construction, there is a component of $B_0 \cap B_1$ which is homeomorphic to the annulus

$$(D_0 - \mathrm{Int}(R_0)) \bigcup_f (D_1 - \mathrm{Int}(R_1)).$$

Further the 3-balls $B_{i,2}$ and $B_{i,3}$ ($i = 0, 1$) are both contained in M_2 (up to homeomorphism). Let now y_i be a point of $\mathrm{Int}(B_{i,2} \cap B_{i,3})$ and let $T_i =$

$N(y_i; M_2)$ be a small regular neighbourhood of y_i in M_2 ($i = 0, 1$). Since $B_{0,j} \cap B_{1,h}$ is empty for each $j, h \geqq 2$, it follows that $T_0 \cap T_1$ is also empty. Let $g: \partial T_0 \to \partial T_1$ be a homeomorphism between two 2-spheres such that $g(\partial T_0 \cap B_{0,j}) = \partial T_1 \cap B_{1,j}$ for each $j = 2, 3$. Now we consider the closed 3-manifold M_3 obtained from $M_2 - (\mathrm{Int}(T_0) \cup \mathrm{Int}(T_1))$ by identifying ∂T_0 with ∂T_1 via g. Then M_3 is homeomorphic to the connected sum $M_2 \# (S^1 \otimes S^2)$ (see [H], lemma 3.8, p. 27). Obviously

$$B_j = (B_{0,j} - \mathrm{Int}(T_0)) \bigcup_g (B_{1,j} - \mathrm{Int}(T_1))$$

is a 3-ball in M_3 since $B_{0,j} \cap B_{1,j}$ is empty for each $j = 2, 3$. Moreover the two 3-balls B_0 and B_1 are both contained in M_3 (up to homeomorphism). Thus M_3 admits a P_0-ball covering $\mathbf{B} = \{B_i / i \in N_3\}$ which is not a P_2-ball covering.

PROPOSITION 5. *Let M^4 be a closed 4-manifold with $H_3(M) = 0$. Then each P_1-ball covering $\mathbf{B} = \{B_i / i \in N_4\}$ of M is a P_2-ball covering of M.*

PROOF. Let (i, j, h, k, r) be a permutation of N_4. The statement is proved if we show the following facts:

(a) $H_1(B_i \cap B_j \cap B_h) = 0$;

(b) $H_1(B_i \cap B_j \cap B_h \cap B_k) = 0$.

The result (a) implies that $H_2(B_i \cap B_j \cap B_h) = 0$, and therefore each component of $B_i \cap B_j \cap B_h$ must be a 2-ball. In fact $B_i \cap B_j \cap B_h$ is a disjoint union of surfaces contained in $B_i \cap B_j = \partial B_i \cap \partial B_j$. If there exists a 2-sphere S^2 in $B_i \cap B_j \cap B_h \subset B_i \cap B_j$ (i.e. $H_2(B_i \cap B_j \cap B_h) \neq 0$), then $S^2 \cap \partial B_h$ must contain a circle so that $H_1(B_i \cap B_j \cap B_h) \neq 0$, a contradiction.

(a) Using the (M, V)-sequence of the pair $(B_i \cap B_j, B_h)$, we get

$$0 = H_2(B_i \cap B_j) \oplus H_2(B_h) \longrightarrow H_2((B_i \cap B_j) \cup B_h)$$
$$\longrightarrow H_1(B_i \cap B_j \cap B_h) \longrightarrow H_1(B_i \cap B_j) \oplus H_1(B_h) = 0,$$

where $H_1(B_i \cap B_j \cap B_h) \simeq H_2((B_i \cap B_j) \cup B_h)$. The (M, V)-sequence of the pair $(B_i \cup B_h, B_j \cup B_h)$ gives

$$0 = H_3(B_i \cup B_h) \oplus H_3(B_j \cup B_h) \longrightarrow H_3(B_i \cup B_j \cup B_h)$$
$$\longrightarrow H_2((B_i \cap B_j) \cup B_h) \longrightarrow H_2(B_i \cup B_h) \oplus H_2(B_j \cup B_h) = 0,$$

since $H_q(B_i \cup B_h) \simeq \tilde{H}_{q-1}(B_i \cap B_h)$ and $B_i \cap B_h$ has 3-balls as connected components. Thus we have

$$H_1(B_i \cap B_j \cap B_h) \simeq H_2((B_i \cap B_j) \cup B_h) \simeq H_3(B_i \cup B_j \cup B_h).$$

Note that the manifold M is orientable since $H_3(M) = 0$ and therefore

$$H_4(M - \mathrm{Int}(B_r)) \simeq H_4(B_i \cup B_j \cup B_h \cup B_k) = 0.$$

Now we prove that $H_3(B_i \cup B_j \cup B_h \cup B_k) = 0$. In fact, using the (M, V)-sequence of the pair $(B_i \cup B_j \cup B_h \cup B_k, B_r)$, we obtain

$$0 = H_4(B_i \cup B_j \cup B_h \cup B_k) \oplus H_4(B_r) \longrightarrow H_4(M)$$
$$\simeq Z \longrightarrow H_3((B_i \cup B_j \cup B_h \cup B_k) \cap B_r)$$
$$= H_3(\partial B_r) \simeq Z \longrightarrow H_3(B_i \cup B_j \cup B_h \cup B_k) \oplus H_3(B_r) \longrightarrow H_3(M) = 0,$$

whence $0 \longrightarrow Z \longrightarrow Z \longrightarrow H_3(B_i \cup B_j \cup B_h \cup B_k) \longrightarrow 0$. Obviously this short sequence implies that $H_3(B_i \cup B_j \cup B_h \cup B_k) = 0$.

Now we prove that $H_3((B_i \cup B_j \cup B_h) \cap B_k) = 0$. Setting

$$X = (B_i \cup B_j \cup B_h) \cap B_k,$$

then X is a (possibly non-connected) 3-manifold with boundary in the 3-sphere ∂B_k. If X_i $(i \in N_m)$ is a component of X, then by the Lefschetz duality we have

$$H_3(X) \simeq H^0(X, \partial X) \simeq FH_0\left(\bigcup_i X_i, \bigcup_i \partial X_i \right) \simeq (m+1) H_0(X_i, \partial X_i) \simeq 0$$

since ∂X_i is non-empty. Using the (M, V)-sequence of the pair $(B_i \cup B_j \cup B_h, B_k)$ we get

$$0 = H_3((B_i \cup B_j \cup B_h) \cap B_k) \longrightarrow H_3(B_i \cup B_j \cup B_h) \oplus H_3(B_k)$$
$$\longrightarrow H_3(B_i \cup B_j \cup B_h \cup B_k) = 0,$$

which implies that $H_3(B_i \cup B_j \cup B_h) \simeq H_1(B_i \cap B_j \cap B_h) = 0$ as requested.

(b) The (M, V)-sequence of the pair $(B_i \cap B_j \cap B_h, B_k)$ gives

$$0 = H_2(B_i \cap B_j \cap B_h) \oplus H_2(B_k) \longrightarrow H_2((B_i \cap B_j \cap B_h) \cup B_k)$$
$$\longrightarrow H_1(B_i \cap B_j \cap B_h \cap B_k) \longrightarrow H_1(B_i \cap B_j \cap B_h) \oplus H_1(B_k) = 0$$

whence $H_1(B_i \cap B_j \cap B_h \cap B_k) \simeq H_2((B_i \cap B_j \cap B_h) \cup B_k)$. Using the (M, V)-sequence of the pair $(B_i \cap B_j, B_k)$ we get

$$0 = H_q(B_i \cap B_j) \oplus H_q(B_k) \longrightarrow H_q((B_i \cap B_j) \cup B_k)$$
$$\longrightarrow H_{q-1}(B_i \cap B_j \cap B_k) = 0$$

for $q = 2, 3$, which implies that

$$H_q((B_i \cap B_j) \cup B_k) = H_q((B_i \cup B_k) \cap (B_j \cup B_k)) = 0 \qquad (q = 2, 3).$$

Since $(B_i \cap B_j \cap B_h) \cup B_k = (B_i \cup B_k) \cap (B_j \cup B_k) \cap (B_h \cup B_k)$, the (M, V)-sequence of the pair $((B_i \cup B_k) \cap (B_j \cup B_k), B_h \cup B_k)$ gives

$$\begin{aligned} 0 = H_3((B_i \cup B_k) \cap (B_j \cup B_k)) \oplus H_3(B_h \cup B_k) \\ \longrightarrow H_3((B_i \cup B_k) \cap (B_j \cup B_k) \cup (B_h \cup B_k)) \\ \longrightarrow H_2((B_i \cap B_j \cap B_h) \cup B_k) \\ \longrightarrow H_2((B_i \cup B_k) \cap (B_j \cup B_k)) \oplus H_2(B_h \cup B_k) = 0, \end{aligned}$$

and therefore

$$\begin{aligned} H_1(B_i \cap B_j \cap B_h \cap B_k) \simeq H_2((B_i \cap B_j \cap B_h) \cup B_k) \\ \simeq H_3((B_i \cup B_k) \cap (B_j \cup B_k) \cup (B_h \cup B_k)) \simeq H_3((B_i \cap B_j) \cup B_h \cup B_k), \end{aligned}$$

where $(B_i \cup B_k) \cap (B_j \cup B_k) \cup (B_h \cup B_k) = (B_i \cap B_j) \cup B_h \cup B_k$. Since

$$(B_i \cap B_j) \cup B_h \cup B_k = (B_i \cup B_h \cup B_k) \cap (B_j \cup B_h \cup B_k),$$

the (M, V)-sequence of the pair $(B_i \cup B_h \cup B_k, B_j \cup B_h \cup B_k)$ gives

$$\begin{aligned} 0 = H_4(B_i \cup B_j \cup B_h \cup B_k) \longrightarrow H_3((B_i \cup B_h \cup B_k) \cap (B_j \cup B_h \cup B_k)) \\ \longrightarrow H_3(B_i \cup B_h \cup B_k) \oplus H_3(B_j \cup B_h \cup B_k) = 0, \end{aligned}$$

whence $H_3((B_i \cup B_h \cup B_k) \cap (B_j \cup B_h \cup B_k)) = 0$. Thus the sequence of isomorphic groups

$$\begin{aligned} H_1(B_i \cap B_j \cap B_h \cap B_k) \simeq H_3((B_i \cup B_k) \cap (B_j \cup B_k) \cup (B_h \cup B_k)) \\ \simeq H_3((B_i \cup B_h \cup B_k) \cap (B_j \cup B_h \cup B_k)) \simeq 0 \end{aligned}$$

concludes the proof. ■

As a direct extension of Proposition 5, we have the following

PROPOSITION 6. *Let M^n be a closed n-manifold with $H_{n-1}(M) = 0$. Let $\mathbf{B} = \{B_i / i \in N_n\}$ be a P_0-ball covering of M such that the components of $\bigcap \{B_j / j \in J\}$ are $(n - h + 1)$-balls for each $J \subseteq N_n$,* card$(J) = h$ *and $2 \leqq j \leqq n - 2$. Then $\mathbf{B}$ is a P_2-ball covering of M.*

4. A characterization theorem

In this section we get a classification of the closed 4-manifolds M^4 with $\mathbf{b}_1(M)=4$. Some results are also proved about closed n-manifolds with P_1-covering number equal to four.

PROPOSITION 7. *Let M^n be a closed orientable n-manifold ($n \geqq 4$) with $\mathbf{b}_1(M)=4$. If $\mathbf{B}=\{B_i/i \in N_3\}$ is a P_1-ball covering of M, then $H_q(M)=0$ ($2 \leqq q \leqq n-2$) and $H_1(M) \simeq H_{n-1}(M) \simeq (r-1)Z$, where r is the number of components of $B_i \cap B_j$ for each $i, j \in N_3$ ($i \neq j$).*

PROOF. Let (i, j, h, k) be a permutation of N_3. The union $B_i \cup B_j$ collapses to the suspension $\Sigma(B_i \cap B_j)$ of $B_i \cap B_j$. Since $B_i \cap B_j$ is a disjoint union of $(n-1)$-balls, $\Sigma(B_i \cap B_j)$ collapses to a graph G formed by two vertices joined by r edges, where r is the number of components of $B_i \cap B_j$. This implies that $B_i \cup B_j \simeq (r-1)_\partial(S^1 \times B^{n-1})$ since $B_i \cup B_j$ is a regular neighbourhood of G in the orientable n-manifold M. Noting that $\partial(B_i \cup B_j)=\partial(B_h \cup B_k)$, we also have $B_h \cup B_k \simeq (r-1)_\partial(S^1 \times B^{n-1})$. The (M, V)-sequence of the pair $(B_i \cup B_j, B_h \cup B_k)$ gives $H_q(M)=0$ ($3 \leqq q \leqq n-2$) if $n>4$ and $H_{n-1}(M) \simeq (r-1)Z$ if $n \geqq 4$. Furthermore, using the exact sequence

$$0=H_2(B_i \cup B_j) \oplus H_2(B_h \cup B_k) \longrightarrow H_2(M) \longrightarrow H_1((r-1)(S^1 \times S^{n-2}))$$

$$\longrightarrow H_1(B_i \cup B_j) \oplus H_1(B_h \cup B_k) \longrightarrow H_1(M) \longrightarrow 0$$

we get $0 \to H_2(M) \to (r-1)Z \to 2(r-1)Z \to H_1(M) \to 0$, whence $H_2(M)$ has no torsion. By the Poincaré duality, we have

$$H_1(M) \simeq H^{n-1}(M) \simeq FH_{n-1}(M) \oplus TH_{n-2}(M) \simeq (r-1)Z$$

since $H_{n-1}(M) \simeq (r-1)Z$, $H_{n-2}(M)=0$ for $n>4$ and $H_2(M)$ has no torsion for $n=4$. Thus the exact sequence

$$0 \to H_2(M) \to (r-1)Z \to 2(r-1)Z \to (r-1)Z \to 0$$

yields $H_2(M)=0$. This completes the proof. ■

PROPOSITION 8.

(a) *Let M^n be a closed simply connected n-manifold. If $\mathbf{b}_1(M) \leqq 4$, then M is trivial.*

(b) *Let M^{2n} be a closed orientable $(2n)$-manifold with Euler characteristic $\chi(M^{2n})=2$. If $\mathbf{b}_1(M) \leqq 4$, then M is trivial.*

PROOF.

(a) For such a manifold M, the sequence of isomorphic groups $\Pi_1(M) \simeq H_1(M) \simeq (r-1)Z \simeq 0$ yields $r = 1$ (see Proposition 7). Thus $B_i \cup B_j \simeq B_h \cup B_k$ is an n-ball and M is the union of two n-balls with common boundary.

(b) Using Proposition 7, we have

$$\chi(M^{2n}) = 1-(r-1)+(-1)^{2n-1}(r-1)+(-1)^{2n} = 4-2r = 2,$$

whence $r = 1$. Now (b) follows as shown in (a). ∎

PROPOSITION 9.

(a) *Let M^4 be a closed 4-manifold. Then* $\mathbf{b}_1(M) = 4$ *iff M is homeomorphic to $k(S^1 \otimes S^3)$ for some integer $k \geqq 1$.*

(b) *If M^4 is a closed non-trivial handle free 4-manifold, then* $\mathbf{b}_1(M) = 5$.

PROOF.

(a) Let $\mathbf{B} = \{B_i / i \in N_3\}$ be a P_1-ball covering of M^4. Noting that

$$B_0 \cup B_1 \simeq B_2 \cup B_3 \simeq (r-1)_\partial(S^1 \otimes B^3),$$

the statement follows from th. 2 of [Mo].

(b) is a direct consequence of (a). ∎

5. Punctured manifolds

In order to prove the following statements we need some definitions.

DEFINITION 3. Let B^n be an n-ball and let $B_1^n, \ldots, B_k^n$ be a finite number of pairwise disjoint n-balls contained in the interior of B. The n-manifold $\mathrm{cl}(B - \bigcup_{i=1}^k B_i; B)$ is called a *k-punctured n-ball*, written $P^n(k)$ (see [H, p. 29]; [KT, p. 139]).

DEFINITION 4. A *k-punctured n-handle* is an n-manifold with boundary obtained by taking the interiors of k disjoint n-manifolds $F_1, \ldots, F_k$ (where $F_i \simeq (r_i)_\partial(S^1 \otimes B^{n-1})$, $i \in N_k^0$) out of the n-manifold $r(S^1 \otimes S^{n-1})$.

If $r_i = 0$ for each $i \in N_k^0$, then we take the interiors of k disjoint n-balls out of $r(S^1 \otimes S^{n-1})$.

If $r_i = 0$ for each $i \in N_k^0$ and $r = 0$, then a k-punctured n-handle is just a $(k-1)$-punctured n-ball in the sense of Definition 3.

Now we prove the following result:

PROPOSITION 10. *Let $P_i^n(k_i)$ be a k_i-punctured n-ball $(i = 1, 2)$ and let $Q_i^{n-1} \subset \partial P_i^n(k_i)$ be a (possibly non-connected) $(n-1)$-manifold such that each* *component of ∂Q_i^{n-1} is an $(n-2)$-sphere bicollared in $\partial P_i^n(k_i)$. If $f: Q_1 \to Q_2$ is a homeomorphism, then the n-manifold $M^n = P_1^n(k_1) \cup_f P_2^n(k_2)$ is a punctured n-handle.*

PROOF. In the present proof each submanifold left unchanged in any quotient space will be denoted by the same symbol. By the generalized Schoënflies theorem (see [R, p. 34]), the components of Q_i^{n-1} $(i = 1, 2)$ are either $(n-1)$-spheres or punctured $(n-1)$-balls. Let

$$W_i = \{S_{i,1}^{n-1}, \ldots, S_{i,p}^{n-1}\} \quad \text{and} \quad V_i = \{P_{i,1}^{n-1}(h_1), \ldots, P_{i,q}^{n-1}(h_q)\}$$

be the set of $(n-1)$-spheres and the set of punctured $(n-1)$-balls in Q_i^{n-1} $(i = 1, 2)$ respectively. Without loss of generality we can always suppose that $f(S_{1,j}^{n-1}) = S_{2,j}^{n-1}$ $(j \in N_p^0)$ and $f(P_{1,r}^{n-1}(h_r)) = P_{2,r}^{n-1}(h_r)$ $(r \in N_q^0)$. If W_1 is non-empty, let f_1 be the restriction of f to $\bigcup W_1$. Then the adjunction space $M_1 = P_1^n(k_1) \cup_{f_1} P_2^n(k_2)$ is the n-manifold obtained by taking the interiors of $k_1 + k_2 - 2p$ disjoint n-balls out of $(p-1)(S^1 \otimes S^{n-1})$. If V_1 is empty, then the proof is completed since $M^n = M_1^n$.

Otherwise, let S_i^{n-1} be the $(n-1)$-sphere of ∂M_1 which contains $P_{i,1}^{n-1}(h_1)$ for $i = 1, 2$. The closure $\mathrm{cl}(S_i^{n-1} - P_{i,1}^{n-1}(h_1); S_i^{n-1})$ is a disjoint union of $h_1 + 1$ $(n-1)$-balls, say $B_{i,0}^{n-1}, \ldots, B_{i,h_1}^{n-1}$. Thus the restriction $f|_{P_{1,1}^{n-1}(h_1)}$ can be extended to a homeomorphism $f_2 : S_1^{n-1} \to S_2^{n-1}$ such that $f_2(B_{1,j}^{n-1}) = B_{2,j}^{n-1}$ for each $j \in N_{h_1}$. Let M_2 be the n-manifold obtained from M_1 by identifying S_1^{n-1} with S_2^{n-1} via f_2. Obviously M_2 is homeomorphic to the manifold obtained by taking the interiors of $k_1 + k_2 - 2p - 2$ disjoint n-balls out of $p(S^1 \otimes S^{n-1})$. Let M_3 be the n-manifold obtained from M_2 by duplicating each n-ball $B_{2,j}^{n-1} = f_2(B_{1,j}^{n-1})$, $j \in N_{h_1}$, into two copies with common boundary. Thus M_3 is homeomorphic to $p(S^1 \otimes S^{n-1})$ minus the interiors of $k_1 + k_2 - 2p + h_1 - 1$ disjoint n-balls. Let now T_i^{n-1} $(i = 1, 2)$ be the $(n-1)$-sphere of ∂M_3 which contains $P_{i,2}^{n-1}(h_2)$. The closure $\mathrm{cl}(T_i^{n-1} - P_{i,2}^{n-1}(h_2); T_i^{n-1})$ is a disjoint union of $h_2 + 1$ $(n-1)$-balls, say $D_{i,0}^{n-1}, \ldots, D_{i,h_2}^{n-1}$ $(i = 1, 2)$. There are two possible cases according to whether or not the $(n-1)$-spheres T_1^{n-1} and T_2^{n-1} coincide.

If $T_1^{n-1} \neq T_2^{n-1}$, i.e. $T_1^{n-1} \cap T_2^{n-1}$ is empty, let $f_3 : T_1^{n-1} \to T_2^{n-1}$ be a homeomorphism which extends the restriction $f|_{P_{1,2}^{n-1}(h_2)}$. The indices can always be chosen so that $f_3(D_{1,j}^{n-1}) = D_{2,j}^{n-1}$ $(j \in N_{h_2})$.

Let M_4 be the n-manifold obtained from M_3 by identifying T_1^{n-1} with T_2^{n-1} via f_3. Thus M_4 is homeomorphic to $(p+1)(S^1 \otimes S^{n-1})$ minus the interiors of $k_1 + k_2 - 2p + h_1 - 3$ disjoint n-balls. Let M_5 be the manifold obtained from M_4 by duplicating each n-ball $D_{2,j}^{n-1} = f_3(D_{1,j}^{n-1})$, $j \in N_{h_2}$, into two copies with common boundary. Obviously M_5 is also obtained by taking the interiors of $k_1 + k_2 - 2p + h_1 + h_2 - 2$ disjoint n-balls out of $(p + 1)(S^1 \otimes S^{n-1})$.

In the other case, i.e. $T_1^{n-1} = T_2^{n-1}$ in ∂M_3, let D_{i,h_2}^{n-1} $(i = 1, 2)$ be the unique $(n-1)$-ball of the closure $\mathrm{cl}(T_1^{n-1} - P_{i,2}^{n-1}(h_2); T_1^{n-1})$ which contains $P_{3-i,2}^{n-1}(h_2)$. The restriction $f|_{P_{1,2}^{n-1}(h_2)}$ can be extended to a homeomorphism

$$f_4 : \mathrm{cl}(T_1^{n-1} - D_{1,h_2}^{n-1}; T_1^{n-1}) \to \mathrm{cl}(T_1^{n-1} - D_{2,h_2}^{n-1}; T_1^{n-1}).$$

By $\tilde{M}_4$ we denote the n-manifold obtained from M_3 by identifying $\mathrm{cl}(T_1^{n-1} - D_{1,h_2}^{n-1}; T_1^{n-1})$ with $\mathrm{cl}(T_1^{n-1} - D_{2,h_2}^{n-1}; T_1^{n-1})$ via f_4. Thus $\tilde{M}_4$ is homeomorphic to $(p + 1)(S^1 \otimes S^{n-1})$ minus the interiors of $k_1 + k_2 - 2p + h_1 - 1$ disjoint n-balls and minus the interior of the submanifold $\lambda_\partial(S^1 \otimes B^{n-1})$ $(\lambda = 1$ for the first step). Here $\lambda_\partial(S^1 \otimes B^{n-1})$ is disjoint from the above-mentioned n-balls, too. Let $\tilde{M}_5$ be the n-manifold obtained from $\tilde{M}_4$ by duplicating each n-ball $D_{2,j}^{n-1} = f_4(D_{1,j}^{n-1})$, $j \in N_{h_2-1}$, into two copies with common boundary. Then $\tilde{M}_5$ is homeomorphic to $(p + 1)(S^1 \otimes S^{n-1})$ minus the interiors of $k_1 + k_2 - 2p + h_1 + h_2 - 1$ disjoint n-balls and minus the interior of $\lambda_\partial(S^1 \otimes B^{n-1})$. Finally, the statement follows by iterating the previous constructions. ∎

Now the following propositions are direct applications of Proposition 10. First we give an alternative simple proof of the main theorem of [KT] and [HM] (see (E), prop. 1, sect. 2). Then we prove some new results for manifolds whose dimensions are greater than three.

STATEMENT (E). PROPOSITION 1. An alternative simple proof.

PROOF. Let M^3 be a closed 3-manifold with $\mathbf{b}_0(M) = 3$. If $\mathbf{B} = \{B_0, B_1, B_2\}$ is a P_0-ball covering of M, then we set

$$W = B_0 \cup B_1 = M - \mathrm{Int}(B_2) \quad \text{and} \quad Q = B_0 \cap B_1 = \partial B_0 \cap \partial B_1.$$

Obviously $\partial W = \partial B_2$ is a 2-sphere and M is the manifold obtained from W by capping off the 2-sphere component of ∂W with a 3-ball. It is very easy to see that the components of the bordered surface Q are punctured 2-balls. Furthermore each 1-sphere of ∂Q is bicollared in ∂B_0 (resp. ∂B_1) since the Schoënflies conjecture is true in dimension $n \leqq 3$. By Proposition 10, W is a punctured 3-handle whose boundary is a 2-sphere, i.e. W is just the manifold obtained by taking the interior of one 3-ball out of $k(S^1 \otimes S^2)$ for some integer $k \geqq 1$. This implies that M^3 is homeomorphic to $k(S^1 \otimes S^2)$ as requested. The reverse implication is trivial since

$$\mathbf{b}_0(S^1 \otimes S^p) = 3 \quad \text{and} \quad \mathbf{b}_0(M_1 \# M_2) \leqq \max\{\mathbf{b}_0(M_1), \mathbf{b}_0(M_2)\}$$

for any two n-manifolds (see also [KT], th. 2.5). ■

PROPOSITION 11. *Let M^4 be a closed 4-manifold with $H_2(M) = 0$. Then $\mathbf{b}_0(M) = 3$ iff M is homeomorphic to $k(S^1 \otimes S^3)$ for some integer $k \geqq 1$.*

PROOF. If $\mathbf{B} = \{B_0, B_1, B_2\}$ is a P_0-ball covering of M^4, then we set $W = B_0 \cup B_1 = M - \mathrm{Int}(B_2)$ and $Q = B_0 \cap B_1 = \partial B_0 \cap \partial B_1$. If we prove that $H_1(\partial Q) = 0$, then the closed surface ∂Q is a disjoint union of 2-spheres imbedded into the 3-sphere ∂B_0. Indeed we have

$$H_1(Q) = H_1(B_0 \cap B_1) \simeq H_2(B_0 \cup B_1) \simeq H_2(W) \simeq H_2(M) = 0.$$

If $P = \mathrm{cl}(\partial B_0 - Q; \partial B_0)$, then the (M, V)-sequence of the pair (P, Q) implies that $H_1(\partial Q) \simeq H_1(P) \oplus H_1(Q) \simeq H_1(P)$. By the Alexander duality (see [Maun], th. 5.3.19), we obtain

$$H_1(P) = H_1(\partial B_0 - \mathrm{Int}(Q)) \simeq H^1(Q) \simeq FH_1(Q) \oplus TH_0(Q) = 0,$$

whence $H_1(\partial Q) = 0$ as claimed. Further the 2-sphere components of ∂Q are bicollared in ∂B_0 by the Schoënflies theorem in dimension $n \leqq 3$. By Proposition 10, it follows that W is a punctured 4-handle whose boundary is a 3-sphere. Thus W is just the manifold obtained by taking the interior of one 4-ball out of $k(S^1 \otimes S^3)$ for some integer $k \geqq 1$. This implies that M^4 is homeomorphic to $k(S^1 \otimes S^3)$ as requested. The sufficient condition is proved as shown in the previous proposition. ■

PROPOSITION 12.

(a) *If M^4 is a closed non-trivial handle free 4-manifold with $H_2(M) = 0$, then $4 \leqq \mathbf{b}_0(M) \leqq \mathbf{b}_1(M) = 5$.*

(b) *Let M^3 be a 3-dimensional homology sphere. Then $\mathbf{b}_0(M^3 \times S^1) = 3$ iff M^3 is trivial.*

(c) *If $\mathrm{cat}(M \times S^1) = \mathbf{b}_0(M \times S^1)$ for any homotopy 3-sphere M, then the 3-dimensional Poincaré conjecture is true.*

PROOF.

(a) Use Propositions 9 and 11.

(b) By the Kunneth formula, we have $H_2(M^3 \times S^1) = H_2(M^3) \oplus H_1(M^3) = 0$ so that the statement follows from Proposition 11.

(c) For a homotopy sphere M, we have $\mathrm{cat}(M) = 2$ and $\mathrm{cat}(M \times S^1) \geqq 3$. Further, the inequality

$$\mathrm{cat}(M \times S^1) \leqq \mathrm{cat}(M) + \mathrm{cat}(S^1) - 1 = 2 + 2 - 1 = 3$$

(see [F, p. 349]) yields $\mathrm{cat}(M \times S^1) = \mathfrak{b}_0(M \times S^1) = 3$. Thus (c) is a consequence of (b). ∎

PROPOSITION 13. *Let M^n be an n-manifold ($n > 7$) with non-empty boundary which is the union of two n-balls B_0, B_1 with disjoint interiors. Suppose that $B_0 \cap B_1 = \partial B_0 \cap \partial B_1$ is an $(n-1)$-manifold whose boundary components are simply connected and bicollared in ∂B_0 (resp. ∂B_1). If $H_q(M) = 0$, $2 \leqq q \leqq n-2$, then M is a punctured n-handle.*

PROOF. With the notation of Proposition 11, let Q be the (possibly non-connected) $(n-1)$-manifold $B_0 \cap B_1$ and $P = \mathrm{cl}(\partial B_0 - Q; \partial B_0)$. Since

$$H_q(Q) = H_q(B_0 \cap B_1) \simeq H_{q+1}(B_0 \cup B_1) \simeq H_{q+1}(M) = 0,$$

$$1 \leqq q \leqq n-3,$$

the (M, V)-sequence of the pair (P, Q) give $H_q(\partial Q) \simeq H_q(P) \oplus H_q(Q) \simeq H_q(P)$ if $1 \leqq q \leqq n-3$. By the Alexander duality, we get

$$H_q(\partial Q) \simeq H_q(P) \simeq H_q(\partial B_0 - \mathrm{Int}(Q))$$

$$\simeq H^{n-q-1}(Q) \simeq FH_{n-q-1}(Q) \oplus TH_{n-q-2}(Q) = 0$$

if $2 \leqq q \leqq n-3$. Now the Hurewicz theorem implies that the components of ∂Q are homotopy $(n-2)$-spheres since they are simply connected and $H_q(\partial Q) = 0$ $(1 \leqq q \leqq n-3)$. For dimension $n - 2 \geqq 5$, these components are just $(n-2)$-spheres which are bicollared in ∂B_0 (resp. ∂B_1) (see [Z], the generalized Poincaré conjecture). As a consequence of Proposition 10, it follows that M is a punctured n-handle. ∎

6. Calculations

In this section we calculate the category and the covering numbers for many closed manifolds. First we recall a classical definition and some well-known results about category of manifolds (see [F], [LS]).

Let M^n be a closed n-manifold and let

$$M_0 \subset M_1 \subset \cdots \subset M_k = M$$

be a sequence of submanifolds of M with dimensions

$$0 \leqq n_0 < n_1 < \cdots < n_k = n$$

respectively. This sequence is said to be an *S-sequence of length $k+1$ in M* if the inclusion induced map

$$M_{n_i-n_{i-1}}(M_i; Z_2) \to H_{n_i-n_{i-1}}(M; Z_2)$$

is a monomorphism for every $i \in N^0_{k-1}$.

If there is an S-sequence of length $k+1$ in a closed n-manifold M^n, then $\mathrm{cat}(M) \geqq k+1$ (see [F], th. 29.3; 29.4, p. 358; [LS], p. 40).

Further, if M_i $(i=1,2)$ is a closed manifold having an S-sequence of length k_i+1, then $\mathrm{cat}(M_1 \times M_2) \leqq k_1+k_2+1$ (see [F], 30.2, p. 359).

PROPOSITION 14. *Let KP^n denote one of the three spaces RP^n (real projective n-space), CP^n (complex projective n-space) or HP^n (quaternionic projective n-space). Then we have*

(1) $\mathrm{cat}(KP^n) = \mathbf{C}(KP^n) = \mathbf{b}_0(KP^n) = n+1$,

(2) $\mathrm{cat}(KP^{n_1} \times \cdots \times KP^{n_s}) = \mathbf{C}(KP^{n_1} \times \cdots \times KP^{n_s})$
$= \mathbf{b}_0(KP^{n_1} \times \cdots \times KP^{n_s}) = n_1 + n_2 + \cdots + n_s + 1$,

(3) $\mathrm{cat}(S^{p_1} \times \cdots \times S^{p_t}) = \mathbf{C}(S^{p_1} \times \cdots \times S^{p_t})$
$= \mathbf{b}_0(S^{p_1} \times \cdots \times S^{p_t}) = t+1$,

(4) $\mathrm{cat}(KP^{n_1} \times \cdots \times KP^{n_s} \times S^{p_1} \times \cdots \times S^{p_t})$
$= \mathbf{C}(KP^{n_1} \times \cdots \times KP^{n_s} \times S^{p_1} \times \cdots \times S^{p_t})$
$= \mathbf{b}_0(KP^{n_1} \times \cdots \times KP^{n_s} \times S^{p_1} \times \cdots \times S^{s_t})$
$= n_1 + n_2 + \cdots + n_s + t + 1$.

PROOF.

(1) Obviously $\mathrm{cat}(KP^n) \geqq n+1$ since KP^n admits an S-sequence

$$KP^0 \subset KP^1 \subset \cdots \subset KP^n$$

of length $n+1$ (compare [F], 31.1, p. 359 for $K=R$). Now RP^n (resp. CP^n, HP^n) is a CW-complex with one cell of dimension i (resp. $2i$, $4i$) for each $i \in N_n$. Thus RP^n (resp. CP^n, HP^n) has a handle decomposition with one handle of index i (resp. $2i$, $4i$) for each $i \in N_n$. By using th. 2.7 of [KT], p. 136, we get $\mathbf{b}_0(KP^n) \leqq n+1$, and therefore (1) is proved.

(2) Since each projective n_i-space KP^{n_i} admits an S-sequence of length n_i+1, we have

$$\mathrm{cat}(KP^{n_1} \times \cdots \times KP^{n_s}) \geqq n_1 + n_2 + \cdots + n_s + 1.$$

Now we recall that

$$\mathbf{b}_0(M_1 \times \cdots \times M_s) \leqq \sum_{i=1}^{s} \mathbf{b}_0(M_i) - s + 1$$

holds for any manifolds $M_1, M_2, \ldots, M_s$. (see [T1], th. 1, p. 113). Thus we have

$$\begin{aligned} n_1 + n_2 + \cdots + n_s + 1 &\leqq \operatorname{cat}(KP^{n_1} \times \cdots \times KP^{n_s}) \\ &\leqq \mathbf{C}(KP^{n_1} \times \cdots \times KP^{n_s}) \leqq \mathbf{b}_0(KP^{n_1} \times \cdots \times KP^{n_s}) \\ &\leqq \sum_{i=1}^{s} \mathbf{b}_0(KP^{n_i}) - s + 1 \\ &= \sum_{i=1}^{s} (n_i + 1) - s + 1 = n_1 + n_2 + \cdots + n_s + 1 \end{aligned}$$

as requested.

(3) See [F], p. 350 and [T1], corollary 1, p. 113.

(4) Use th. 1 [T1], p. 113 and recall that KP^{n_i} (resp. S^{p_j}) has an S-sequence of length $n_i + 1$ (resp. 2). ∎

PROPOSITION 15.

(a) *Let M^4 be a smooth non-trivial simply connected closed 4-manifold. Then there exist integers k and h such that*

$$\begin{aligned} \operatorname{cat}(M \# k(CP^2) \# h(-CP^2)) &= \mathbf{C}(M \# k(CP^2) \# h(-CP^2)) \\ &= \mathbf{b}_0(M \# k(CP^2) \# h(-CP^2)) = 3. \end{aligned}$$

(b) *If V is a non-trivial non-singular hypersurface of CP^3, then*

$$\operatorname{cat}(V) = \mathbf{C}(V) = \mathbf{b}_0(V) = 3.$$

PROOF.

(a) For such a manifold M, there exist integers k and h such that $M^4 \# k(CP^2) \# h(-CP^2)$ is a connected sum whose factors are either CP^2 or $-CP^2$ (see [Maund], corollary 1.6). Now use Proposition 14 and th. 2.5 of [KT] to obtain the assertion.

(b) It is well-known that V admits a special handlebody decomposition with one handle of index 0, $\beta_2(V) \neq 0$ handles of index 2 and one handle of index 4 (see [Mand], p. 59). Here $\beta_2(V)$ denotes the 2nd Betti number of V. Thus the result is proved by using th. 2.7 [KT], p. 136. ∎

PROPOSITION 16. *Let F and F' be non-trivial closed (orientable or not) surfaces. Then we have*

(1) $\mathrm{cat}(F \times F') = \mathbf{C}(F \times F') = \mathbf{b}_0(F \times F') = \mathbf{b}_1(F \times F') = 5$,
(2) $\mathrm{cat}(F \times S^2) = \mathbf{C}(F \times S^2) = \mathbf{b}_0(F \times S^2) = 4$ *and* $\mathbf{b}_1(F \times S^2) = 5$.

Proof. If F is an orientable (resp. non-orientable) surface of genus g (resp. h), then $H_1(F; Z_2) \simeq 2gZ_2$ (resp. $H_1(F; Z_2) \simeq hZ_2$).

Let F_1 be a circle in F representing an element of a base of $H_1(F; Z_2)$ and let F_0 be a point of F_1. It is very easy to see that $F_0 \subset F_1 \subset F_2 = F$ is an S-sequence in F and therefore we have

$$5 \leqq \mathrm{cat}(F \times F') \leqq \mathbf{C}(F \times F') \leqq \mathbf{b}_0(F \times F')$$
$$\leqq \mathbf{b}_0(F) + \mathbf{b}_0(F') - 1 = 5$$

as requested.

Noting that the 2-sphere S^2 has an S-sequence of length two, it follows that

$$4 \leqq \mathrm{cat}(F \times S^2) \leqq \mathbf{C}(F \times S^2) \leqq \mathbf{b}_0(F \times S^2)$$
$$\leqq \mathbf{b}_0(F) + \mathbf{b}_0(S^2) - 1 = 4$$

Finally $\mathbf{b}_1(F \times S^2) = 5$ since $H_2(F \times S^2) = H_2(F) \oplus H_0(F)$ is not null (also compare Proposition 7). ∎

Proposition 17. *Let M^{2n} be a smooth non-trivial $(n-1)$-connected closed $(2n)$-manifold ($n \geqq 3$). Then we have*

$$\mathrm{cat}(M) = \mathbf{C}(M) = \mathbf{b}_0(M) = 3.$$

Further, if $\mathbf{B} = \{B_0, B_1, B_2\}$ is a P_0-ball covering of M, then $H_q(B_0 \cap B_1 \cap B_2) \simeq H_q(B_i \cap B_j) = 0$ for $1 \leqq q \leqq n-2$ $(i \neq j)$, $H_{n-1}(B_i \cap B_j) \simeq \Pi_n(M)$ and $H_{n-1}(B_0 \cap B_1 \cap B_2) \simeq \Pi_n(M) \oplus \Pi_n(M)$.

Proof. Following [Sm], p. 39, let $\mathbf{H}(2n, k, n)$ be the set of manifolds obtained from a $(2n)$-ball B^{2n} by attaching k handles of index n. As proved in [Sm], th. 1.1, the manifold M with a $(2n)$-ball removed belongs to $\mathbf{H}(n) = \bigcup_{k=0}^{\infty} \mathbf{H}(2n, k, n)$.

By th. 2.7 [KT], we get

$$\mathrm{cat}(M) = \mathbf{C}(M) = \mathbf{b}_0(M) = 3.$$

Let now (i, j, h) be a permutation of N_2. The (M, V)-sequence of the pair $(B_i \cup B_j, B_h)$ gives $H_q(B_i \cup B_j) \simeq H_{q-1}(B_i \cap B_j) \simeq 0$ if $2 \leqq q \leqq n-1$ and $H_n(B_i \cup B_j) \simeq H_{n-1}(B_i \cap B_j) \simeq H_n(M) \simeq \Pi_n(M)$ by the Hurewicz theorem.

The (M, V)-sequence of the pair $(B_0 \cap B_2, B_1 \cap B_2)$ implies that $H_{q-1}(B_0 \cap B_1 \cap B_2) \simeq 0$ if $2 \leqq q \leqq n-1$ and

$$H_{n-1}(B_0 \cap B_1 \cap B_2) \simeq H_{n-1}(B_0 \cap B_2) \oplus H_{n-1}(B_1 \cap B_2) \simeq \Pi_n(M) \oplus \Pi_n(M)$$

as requested. ■

Now we conclude this section with some applications about the category of closed 3-manifolds. In particular, as a direct consequence of our results, we give an alternative simple proof of th. 5.1 [Si], p. 409 (compare Proposition 20, point (1)).

PROPOSITION 18. *Let M^3 be a closed 3-manifold. Then* cat$(M) = 3$ *iff M is homeomorphic to $\Sigma \# k(S^1 \otimes S^2)$ for some integer $k \geqq 1$ where Σ is a homotopy 3-sphere.*

PROOF. As proved in [Ta], prop. 2.4, p. 201, if cat$(M) = 3$, then M admits a Lusternik–Schnirelmann filling $\{Q_0, Q_1, Q_2\}$, that is, a covering of M with bordered contractible 3-manifolds whose interiors do not intersect.

Obviously ∂Q_i $(i = 0, 1, 2)$ is the 2-sphere since Q_i is a contractible 3-manifold with non-empty boundary. Thus the Van Kampen theorem implies that $\Pi_1(M) = \Pi_1(Q_0) * \Pi_1(Q_1 \cup Q_2) \simeq \Pi_1(Q_1 \cup Q_2)$.

Since Q_1 and Q_2 are contractible, the group $\Pi_1(M) \simeq \Pi_1(Q_1 \cup Q_2)$ is just a free group whose rank is given by the number of connected components of $Q_1 \cap Q_2$ minus one (use the Van Kampen theorem too). Then M is homeomorphic to the connected sum $\Sigma \# k(S^1 \otimes S^2)$ for some integer $k \geqq 1$, where Σ is a homotopy 3-sphere (see [H], th. 5.3, p. 57). Now the proof is completed since the converse implication is trivial. ■

PROPOSITION 19.

(1) *If M^3 is a closed handle free 3-manifold which contains no fake 3-cells, then* cat$(M) = 4$.
(2) *If M^3 is a large Seifert manifold, then* cat$(M) = 4$.
(3) *If $L(p, q)$ is the lens space of type (p, q), $p > 1$, then* cat$(L(p, q)) = 4$.

PROOF. The proofs are direct consequences of Proposition 18. ■

PROPOSITION 20.

(1) *If M^3 is a closed 3-manifold with $H_1(M) \neq 0$ and $H_2(M) = 0$, then* cat$(M) = 4$ (see th. 5.1 [Si], p. 409).
(2) *If M^3 is a closed 3-manifold with $H_1(M) = 0$ and $H_2(M) \neq 0$, then* cat$(M) = 4$.

(3) *If* M^3 *is a homology* 3-*sphere with* $\Pi_1(M) \neq 0$, *then* $\mathrm{cat}(M) = 4$. *In particular, the category of the Poincaré manifold is four.*

PROOF.

(1) Suppose that $\mathrm{cat}(M) \leqq 3$. Then $\mathrm{cat}(M) = 3$ since M is not a homotopy sphere. If $H_2(M) = 0$, then M is orientable, whence $H_1(M) \simeq H^2(M) \simeq FH_2(M) \oplus TH_1(M) \simeq TH_1(M) \neq 0$, a contradiction since $H_1(M)$ is a free group whenever $\mathrm{cat}(M) = 3$.

(2) As shown in (1), suppose that $\mathrm{cat}(M) = 3$, i.e. $M \simeq \Sigma \# k(S^1 \otimes S^2)$ for some integer $k \geqq 1$. Then $M \simeq \Sigma$ as $H_1(M) = 0$, whence $\mathrm{cat}(M) = 2$, a contradiction.

(3) The proof is similar to the one given in (2). ■

PROPOSITION 21. *For any closed* 3-*manifold* M^3 *with* $\mathrm{cat}(M) = 4$ *we have*

$$\mathrm{cat}(M \times S^1) = \mathbf{C}(M \times S^1) = \mathbf{b}_0(M \times S^1) = \mathbf{b}_1(M \times S^1) = 5.$$

PROOF. Using corollary (6.7) [Si], p. 412, we get $\mathrm{cat}(M \times S^1) = \mathrm{cat}(M) + 1 = 5$ since $\mathrm{cat}(M) = 4 \geqq (\dim(M) + 5)/2$. This implies the statement. ■

PROPOSITION 22.

(1) *If* RP^2 *is a real projective plane imbedded in a closed* 3-*manifold* M, *then* $\mathrm{cat}(M) = 4$.

(2) *If* M, M' *are closed* 3-*manifolds which contain real projective planes, then*

$$\mathrm{cat}(M \times M') = \mathbf{C}(M \times M') = \mathbf{b}_0(M \times M') = \mathbf{b}_1(M \times M') = 7.$$

PROOF.

(1) By lemma 5.1 [H], p. 56, the inclusion induced map $H_1(RP^2; Z_2) \to H_1(M; Z_2)$ is a monomorphism.

Let M_1 be a circle in RP^2 representing the generator of $H_1(RP^2; Z_2) \simeq Z_2$ and let M_0 be a point of M_1.

Obviously $M_0 \subset M_1 \subset M_2 = RP^2 \subset M_3 = M$ is an S-sequence of length four in M, whence $\mathrm{cat}(M) = 4$.

(2) Since M and M' have S-sequences of length four, it follows that $\mathrm{cat}(M \times M') \geqq 7$, as claimed. ■

7. Critical points of a Morse function

In this section we prove some results about the minimum number of critical points (resp. values) of Morse functions on a closed smooth manifold.

As general references about Morse theory we refer to [Maz], [Mil], [MC]. Here we recall some definitions and notations.

Let M^n be a closed smooth n-manifold and let $f: M \longrightarrow R$ be a smooth real-valued function defined on M. A point $p \in M$ is said to be a *critical point* of f if and only if $(\mathbf{d}f)_p = 0$, i.e.

$$\left(\frac{\partial f}{\partial x^1}\right)_p = \cdots = \left(\frac{\partial f}{\partial x^n}\right)_p = 0$$

with respect to a local coordinate system $(x_1, \ldots, x_n)$ around p. The real number $f(p)$ is called a *critical value* of the function f.

A critical point p is said to be *non-degenerate* iff the matrix

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_p$$

is non-singular. It is well-known that non-degeneracy does not depend on the local coordinate system around p. Further non-degenerate critical points are isolated (see [Mil], corollary 2.3, p. 8).

A smooth function $f: M \longrightarrow R$ is called a *Morse function* iff all its critical points are non-degenerate.

By the compactness of M, the number of critical values (resp. points) of f, written $\mu_M(f)$ (resp. $\mathbf{F}_M(f)$), is finite.

Following [Mi], p. 49 (resp. [Ta], p. 198), let $\mu(M)$ (resp. $\mathbf{F}(M)$) be the minimum, over all Morse functions f on M, of the integer numbers $\mu_M(f)$ (resp. $\mathbf{F}_M(f)$), i.e.

$$\mu(M) = \min\{\mu_M(f)/f: M \longrightarrow R \text{ is a Morse function}\},$$

$$\mathbf{F}(M) = \min\{\mathbf{F}_M(f)/f: M \longrightarrow R \text{ is a Morse function}\}.$$

The following results are well-known:

PROPOSITION 23.

(1) REEB THEOREM. *Let M^n be a closed smooth n-manifold. Then $\mu(M) = 2$ (resp. $\mathbf{F}(M) = 2$) iff M is trivial* ([Mil], th. 4.1, p. 25).

(2) *If M^n is a closed smooth n-manifold, then*

$$\operatorname{cat}(M) \leqq \mathbf{C}(M) \leqq \mu(M) \leqq n + 1$$

([Mi], th. 1, p. 49).

(3) *Let M^3 be a closed smooth 3-manifold. Then $\mu(M) = 3$ (resp. $\mathbf{F}(M) = 3$)*

iff M is homeomorphic to $k(S^1 \otimes S^2)$ for some integer $k \geqq 1$ ([Ta], 3.3, p. 208).

In order to prove the main result of this section, we need the following lemma:

LEMMA 24. *Let M^n be an n-manifold with non-empty boundary and $\mathbf{b}_0(M) \leqq k$. If $\tilde{M}^n$ is an n-manifold obtained from M by attaching mutually disjoint handles of various indices on ∂M, then $\mathbf{b}_0(\tilde{M}) \leqq k+1$.*

PROOF. The statement is a simple consequence of th. 2.7 [KT]. Here we include the proof to make the reading clear.

Let $\mathbf{B} = \{B_1, B_2, \ldots, B_t\}$ be a P_0-ball covering of M such that card($\mathbf{B}$) = $\mathbf{b}_0(M) = t \leqq k$. Suppose that $\tilde{M}$ is obtained from M by attaching r mutually disjoint handles

$$H_1^{p_1}, H_2^{p_1}, \ldots, H_r^{p_r}$$

of (possibly different) indices $p_1, p_2, \ldots, p_r$, respectively.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{t-1}$ be pairwise disjoint proper arcs in the closure

$$\mathrm{cl}\left(\dot{\bigcup_{i=1}^{t}} \partial B_i - \dot{\bigcup_{j=1}^{r}} H_j^{p_j} \right)$$

such that the union

$$\bigcup_{i=1}^{t-1} \alpha_i \cup \dot{\bigcup_{j=1}^{r}} H_j^{p_j}$$

is connected.

If $N(\bigcup_{i=1}^{t-1} \alpha_i; M)$ is a regular neighbourhood of the union $\bigcup_{i=1}^{t-1} \alpha_i$ in M, then

$$B_0' = N\left(\dot{\bigcup_{i=1}^{t-1}} \alpha_i;\ M \right) \cup \dot{\bigcup_{j=1}^{r}} H_j^{p_j}$$

is an n-ball in $\tilde{M}$. Now we define $B_i' = \mathrm{cl}(B_i - B_0')$ for each $i \in N_t^0$.

By construction each B_i' is an n-ball in $\tilde{M}$. Moreover the collection $\mathbf{B}' = \{B_i' / i \in N_t\}$ is a P_0-ball covering of $\tilde{M}$. Now the proof is completed since $\mathbf{b}_0(\tilde{M}) \leq \mathrm{card}(\mathbf{B}') = t+1 \leqq k+1$. ■

PROPOSITION 25. *Let M be a closed smooth n-dimensional manifold. Then we have*

$$\mathrm{cat}(M) \leqq \mathbf{C}(M) \leqq \mathbf{b}_0(M) \leqq \mu(M) \leqq \mathbf{F}(M) \leqq n+1.$$

PROOF. The sequence of inequalities $\mathrm{cat}(M) \leqq \mathbf{C}(M) \leqq \mu(M) \leqq n+1$ was proved in [Mi], th. 1, p. 49. Furthermore $\mathbf{F}(M) \leqq n+1$ as shown in [Ta], p. 198. By definition the inequality $\mathbf{C}(M) \leqq \mathbf{b}_0(M)$ is trivial.

In order to prove that $\mu(M) \leqq \mathbf{F}(M)$, let $p_0, p_1, \ldots, p_r$ be the critical points of a Morse function f on M such that $\mathbf{F}(M) = \mathbf{F}_M(f) = r+1$.

If $c_0 < c_1 < \cdots < c_t$ are the critical values of f, then we have $t \leqq r$, whence $\mu(M) \leqq t+1 \leqq r+1 = \mathbf{F}(M)$ as requested.

Now it remains to prove that $\mathbf{b}_0(M) \leqq \mu(M)$.

The proof is obtained by means of a simple modification of the method described in the proof of th. 1.1, [Mi]. Here we use also the previous Lemma 24. Following [Mi], th. 1.1, let $c_0 < c_1 < \cdots < c_t$ be the critical values of a Morse function f on M such that $\mu(M) = \mu_M(f) = t+1$. We can always assume that f has only one critical point p with critical value c_0. The index of p is zero (see [Mi], p. 50).

Recall that a non-degenerate critical point p is said to have *index* λ iff for some local coordinate system $(x_1, x_2, \ldots, x_n)$ around p we have

$$f(x_1, \ldots, x_n) = f(p) - (x_1)^2 - \cdots - (x_\lambda)^2 + (x_{\lambda+1})^2 + \cdots + (x_n)^2$$

where $\lambda \in N_n$ and $p = (0, \ldots, 0)$.

If $c_0 < a < c_1$, then $M^a = f^{-1}((-\infty, a])$ is an n-ball in M, whence $\mathbf{b}_0(M^a) = 1$. By induction, we suppose that $\mathbf{b}_0(M^b) \leqq k$ whenever $c_{k-1} < b < c_k$ and $1 \leqq k \leqq t-1$. The statement will be proved once we have shown that $\mathbf{b}_0(M^d) \leqq k+1$ if $c_k < d < c_{k+1}$.

Indeed we have $\mathbf{b}_0(M) \leqq t+1 = \mu(M)$ in the last step.

By [Mil], th. 3.1, th. 3.2, p. 14, Remark p. 17, [Mi], th. 1, p. 50, the manifold M^d is obtained from M^b by attaching mutually disjoint handles of various indices.

In fact, let $p_1, p_2, \ldots, p_r$ be the critical points of f with critical value c_k. It is well-known that passing a critical point p_i of index λ_i corresponds to attaching a handle

$$H_i^{\lambda_i} = B^{\lambda_i} \times B^{n-\lambda_i}$$

of index λ_i to M^b (see the quoted papers). These handles of various indices can be assumed to be pairwise disjoint. By Lemma 24 and the inductive hypothesis $\mathbf{b}_0(M^b) \leqq k$, it follows that $\mathbf{b}_0(M^d) \leqq k+1$. This concludes the proof. ■

By using Proposition 25, we can directly obtain some bounds or the exact values of the invariants $\mu(M)$ and $\mathbf{F}(M)$ for any closed smooth manifold M which satisfies any of the statements listed in Section 6.

Now we conclude this section with some applications of Proposition 25. First we give an alternative simple proof of th. 3.3 [Ta], p. 208 (also compare Proposition 23, point (3)). Finally, the following Proposition 26 is an immediate consequence of Proposition 25 and of [Mi] (th. 2, th. 3, th. 6, lemma 4.1).

STATEMENT (3). PROPOSITION 23. An alternative simple proof.

PROOF. If $\mathbf{F}(M) = 3$ (resp. $\mu(M) = 3.$), then $\mathbf{b}_0(M) \leqq 3$ (see Proposition 25). If $\mathbf{b}_0(M) = 2$, then M is the 3-sphere so that the Reeb theorem gives

$$\mathbf{F}(M) = \mu(M) = 2,$$

a contradiction. Thus the result $\mathbf{b}_0(M) = 3$ implies that M is homeomorphic to $k(S^1 \otimes S^2)$ for some integer $k \geqq 1$ (see Proposition 1, point (e)). ■

PROPOSITION 26.

(1) *Let M^n be a closed smooth n-manifold ($n \geqq 5$). If M is simply connected, then*

$$\operatorname{cat}(M) \leqq \mathbf{C}(M) \leqq \mathbf{b}_0(M) \leqq \mu(M) \leqq n - 1.$$

(2) *Let M_1^n be an orientable closed smooth n-manifold ($n \geqq 3$). Then M_1 cobounds a closed smooth n-manifold M_2 with*

$$\operatorname{cat}(M_2) \leqq \mathbf{C}(M_2) \leqq \mathbf{b}_0(M_2) \leqq \mu(M_2) \leqq n - 1.$$

(3) *Let M^n be a closed smooth n-manifold. If there exists an S-sequence*

$$M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with length $n + 1$ and $\dim(M_i) = i \in N_n$, *then*

$$\operatorname{cat}(M) = \mathbf{C}(M) = \mathbf{b}_0(M) = \mathbf{b}_1(M) = \mu(M) = \mathbf{F}(M) = n + 1.$$

(4) *Let M^n be a closed smooth n-manifold. If there exists $x \in H^1(M; Z_2)$ with cup product $x^n \neq 0$ ($x^n = x \cup \cdots \cup x$, n times), then*

$$\mathrm{cat}(M) = \mathbf{C}(M) = \mathbf{b}_0(M) = \mathbf{b}_1(M) = \mu(M) = \mathbf{F}(M) = n + 1.$$

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